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Structuring Probability Assessments

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A probability assessment framework is outlined that enables decision-makers to determine a probability distribution over possible events or scenarios they could face in the future.The methodology of the analytic hierarchy process can be utilized in the procedures. Bayesian revision accounting for new developments can be used to calculate posterior probabilities using the same procedures.

*Keywords:*decision theory, probability assessment, analytic hierarchy process, Bayesian revision

Introduction

In facing decisions involving uncertain future events, people (hereafter the decision-maker or DM) possess judgments (however tentative) as to the relative likelihoods of the possible events or scenarios. If they knew for sure which event/scenario was going to occur, the action to undertake now (sell car, house, etc.) would be evident. And this action could be undertaken even if the event in question was only “almost certain”. But when several events all reflect moderate or similar probabilities, the best action to implement is not so clear.

To make the best decision, the DM should construct a structure around the possible events leading to analysis that would more clearly indicate the “optimal” action (e.g., sell, buy, wait) to undertake now. This means accurate assessment of event probabilities. First, he/she can order the relevant events from least to most likely with some events possibly exhibiting equal likelihood. The harder task is to now assign numbers reflecting how much “more likely” one event is than another. The pairwise judgments can be likened to appraising two teams or individuals competing in a tournament. The pairwise value arrived at can then be considered as a pre-game estimate of the score. Although it is not to be expected that the estimate is 100% accurate, it would be expected to predict the winner or a draw.

This pairwise comparison procedure forms the basis for a decision methodology articulated by Thomas L. Saaty (1980)called the Analytic Hierarchy Process or AHP. He (2005)later expanded this structure into the Analytic Network Process or ANP. Saaty’s methodology encompassed the complete decision process leading to the choice of an optimal act or decision to implement. Here we restrict ourselves to using this methodology for the task of estimating probabilities for the possible events that will impact on the action taken as in Hughes (2009; 2010). A modern review of AHP methodology is presented by Brunelli (2015).

Creating a Structure for Assessing Probabilities

Recall the events have been ordered from least to most likely. Taking the first two events in the ordering the structure requires a judgment as to the extent to which the more likely event (the second event) exceeds the

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likelihood of the first event. Normalising the first event at 1.0, we can state that equal likelihood for the second event requires the same value 1.0. But what if the second event's likelihood is greater? If it were twice as likely, the number 2.0 reflects this as in 1.0×2. If it is less than twice as likely in the DM's judgment, a number between 1.0 and 2.0 would be appropriate—the greater the value, the greater the relative likelihood of the second event over the first. Note that a 1-in-100 year flood is twice as likely as a 1-in-200 year flood (ignoring start times). Similarly, a re-run of history three times for an event to occur makes that event 4/3 times "more likely" than an event requiring a four-time re-run.If available, odds judgments can motivate the choice of the correct value. For example, a 3:1 event has a 25% chance of occurrence versus a 20% chance for a 4:1 event. Accordingly, the 3:1 event is 25% "more likely" than the 4:1 event (i.e., 5/4 or 1.25) and twice as likely as a 7:1 event (8/4). Choosing an appropriate "more likely" value may be easier for most DMs (with practice) than setting odds. Odds formation is typically based on a history of similar events whereas the situation for the DM will usually be unique to him/her.

A nomenclature or taxonomy mapping qualitative "more likely" judgments (the primary judgments) into an appropriate value is detailed in Table 1.

Table 1

Nomenclature or Taxonomy for Assigning "More Likely" Values Between 1.0 and 2.0

More likely pairwise judgment of the DM	Value range	Mid-point
Equal likelihood	1.0	1.0
Small or slight increase in likelihood	1.0^{+}-1.2^{-}	1.10
Moderate or significant increase in likelihood	1.2-1.80^{-}	1.50
Large or substantial increase in likelihood	1.80-2.0^{-}	1.90
Twice as likely	2.0	2.0

For "more likely" values exceeding 2.0, the same procedure can be followed where 3.5 would indicate an event whose likelihood is between three and four times more likely (a somewhat imprecise judgment but all that might be possible) than its less likely pairwise alternative. Note that persistent over-optimism (pessimism) in the likelihood judgments may cancel out in the series of *relative* pairwise assessments leading to a "correct" initial probability distribution.

To illustrate the calculation of probabilities, a three-event scenario is detailed in Table 2. Note that A is the least likely event and C the most likely event in the ordering.

Table 2

Probability Derivation for aThree-Event Scenario Where Events Have Similar Likelihood

Event order	Pairwise more likely value	Compound likelihood	Probability	Percent probability
A	1.00	1.00	1.00/4.4375 = 0.225	23
B	1.25	1.00×1.25 = 1.25	1.25/4.4375 = 0.282	28
C	1.75	1.00×1.25×1.75 = 2.1875	2.1875/4.4375 = 0.493	49
		4.4375	**1.000**	**100**

The pairwise "more likely" values in Table 2 reflect the judgments that B is 25% "more likely" than A and C is 75% "more likely" than B. That is, both B and C are significantly "more likely" than their less likely alternative but to differing degrees.

Given the rudimentary or imprecise nature of the pairwise "more likely" judgments, a percentage probability seems appropriate for the final probability judgments in the last column of Table 2. Of course, the DM looking at the above may report the probabilities for A, B, and C respectively at 20%, 30%, and 50%. This could reflect a motivation to avoid any criticism of spurious accuracy. Alternatively, the DM may judge, on reflection, that the difference in probability between A and B to be of the order of 10% and not 5% as in the original calculations.

In the reported calculations, we see that 50%/30% or 1.67 is less than the 1.75 used in Table 2. The initial "more likely" judgments or primary judgments may be rudimentary or tentative. Their main objective, however, is to establish the "ballpark" within which the DM can make his/her final assessments after full consideration of the resulting initial probability distribution as in Table 2.

If there were stated market odds at 3:1 for Event B (probability 0.25), this judgment could be incorporated into the procedure using appropriate values for the B over A judgment (i.e., BA) and the C over B judgment (i.e., CB). Alternatively, the DM may decide that B is a 25% chance at the start of the process.In either case, the constraint becomes:

$$BA = 1/(3 - CB)$$

Accordingly, if C is judged to be between twice or three times "more likely" than B (pairwise value 2.5) then BA is necessarily equal to 2.0 and the resulting probability distribution over A, B, and C respectively becomes 0.125, 0.25, and 0.625.

Note too that if a judgment has one event say 10 times "more likely" than the alternative, the probability for the alternative (and all preceding events in the ordering) will be very low. In general, the larger (smaller) the latter pairwise values are, the smaller (larger) the resulting probabilities for the preceding less likely events in the ordering are.

A complete AHP analysis for the above necessitates an additional C over A pairwise judgment and a resulting 3×3 reciprocal matrix as in Brunelli (2015).Consistency with the previous pairwise judgments in Table 2 would require this value to be 2.1875 (or 35/16). If, however, a "rudimentary or tentative" value of 2 were utilized (i.e., C is twice as likely as A), the principal eigenvector procedure results in a probability distribution over A, B, and C respectively of 0.234, 0.284, and 0.482,only marginally different from the distribution in Column 4 of Table 2. Since 2 is slightly less than the perfectly consistent 2.1875 pairwise value, the probability of A goes up slightly and that for C down slightly with P(B) virtually unchanged.

Table 3 details differing probability distributions for changing values in the C over A judgment. The results show that for a wide range in "more likely" values (i.e., 2 to 3 for CA), resulting probabilities do not change dramatically. The maximum range is ±5% depending on the CA values selected.

Table 3

Probabilities for Differing C Over A Values

Events	CA = 2.0	CA = 35/16	CA = 2.5	CA = 2.75	CA = 3.0	Range	Final
A	0.233 981	0.225 352	0.212 853	0.204 202	0.196 502	0.23-0.20	0.22
B	0.283 868	0.281 690	0.278 178	0.275 485	0.272 900	0.28-0.27	0.28
C	0.482 151	0.492 958	0.508 969	0.520 313	0.530 598	0.48-0.53	0.50
Total	**1.000 000**	**1.000 000**	**1.000 000**	**1.000 000**	**1.000 000**		**1.00**

Table 3 shows that if the DM were unsure of the pairwise value for CA in the 2-3 range, the final values would be feasible probabilities based on all results. Alternatively, using the CA value of 2.5 gives almost identical percentage probabilities. Although the additional judgments between non-adjacent events, such as CA may improve the "correctness" of the initial probability distribution via accurate pairwise values, they could also prove redundant if they are (almost) perfectly consistent with the initial adjacent event pairwise judgments. The CA values equal to 2 or 35/16 illustrate this in Table 3.

An alternative to the principal eigenvector method for deriving the probabilities is the geometric mean procedure of Crawford (1987). The geometric mean of each row of the reciprocal matrix is calculated and then normalized to sum to unity. Hughes (2009) used both procedures for a 9× 9 matrix and the maximum difference in absolute value was 0.0031, which would only affect resulting percentage probabilities by at most 1% if at all. Other analyses suggest a difference of up to 2% in percentage probabilities is possible with both methods.

It should also be clear that not all information on the probabilities is necessarily captured in a series of "rudimentary or tentative" pairwise judgments. Other information, such as P(B) is higher than P(A) by at least 10% or Event C is assuredly less than a 50% chance will play a legitimate role in the DM's determination of the final probability distribution.

Bayesian Revision

Bayesian revision allows us to update current probabilities in the light of "news" or information that arrives subsequently to the probabilities derived in Table 2. The Table 2 probabilities are the prior probabilities denoted as P'(A), etc. in Bayesian terminology. Subsequent to the "news" arrival, we desire to derive P(A│"news") or P"(A) as well as P"(B) and P"(C) or the posterior probabilities. To illustrate the derivation, suppose the "news" in this case favours B most (hence B "most consistent" with the "news") with A least consistent and C with intermediate consistency. Again, "more consistent" judgments are now required on the same basis as those for the "more likely" judgments above when deriving the prior distribution. Hence 1.0 is for the least consistent event and then pairwise consistent judgments successively up to the most consistent event.Illustrative judgments and likelihoods are shown below starting with 1.0 for Event A. Note that the event ordering for consistency with the "news" is now A, C, and finally B (most consistent).Table 4 details the likelihood calculations for the three events in this illustrative example.

Table 4

Likelihood Derivations for the Possible Events After the "News" Becomes Known

Event order	Pairwise more consistent value	Compound likelihood	Likelihood	Percent likelihood
A	1.00	1.00	1.00/7 = 0.143	14
C	2.00	1.00×2.00 = 2.00	2.00/7 = 0.286	29
B	2.00	1.00×2.00×2.00 = 4.00	4.00/7 = 0.571	57
		7.00	**1.000**	**100**

Table 4 shows Event B with a likelihood of 57% which is almost twice as consistent with the "news" as is Event C at 29%. Similarly, Event C is slightly more than twice as consistent as Event A and all values accord closely enough with the illustrative primary judgments in Table 4. Note that if the primary judgments in Table 4 were respectively 1.0, 1.90 (i.e., C with a smaller likelihood gain over A than previously), and 2.15 (i.e., B with a greater likelihood gain over C than previously), the resulting likelihoods for A, C, and B would be respectively 14%, 27%, and 59%.

In some cases, events will have zero consistency with the “news” leading to a likelihood value of zero. This means posterior probabilities for these events will be zero and they can be eliminated from further analysis. For our illustrative “more consistent” judgments in Table 4, Table 5 summarizes the calculation of the posterior probabilities following the “news”.

Table 5

Derivation of the Posterior Probabilities Following the “News”

Event	Prior	Likelihood	Joint	Posterior	Posterior percent
A	0.225	0.143	0.225×0.143 = 0.032	0.032/0.334 = 0.096	10
B	0.282	0.571	0.282×0.571 = 0.161	0.161/0.334 = 0.482	48
C	0.493	0.286	0.493×0.286 = 0.141	0.141/0.334 = 0.422	42
	1.000	**1.000**	**0.334**	**1.000**	**100**

In this illustrative example, the “news” has promoted the probability of Event B above that of C, although only a difference of 6% results. Again, the DM may report final probabilities for A, B, and C respectively at say 10%, 50%, and 40% to accord with generally accepted sensibilities on probability accuracy. Alternatively, the DM may decide that equal likelihood at 45% for both B and C is the best assessment in the “ballpark” based on all information to date. If the alternative likelihoods above are utilized, the resulting posterior probabilities for A, B, and C respectively would be 10%, 50%, and 40%.

Rationale for the Pairwise Comparisons of Likelihoods

The direct estimation of event probabilities one at a time is a flawed process. The assessment of the probability of the last event considered is predetermined as one minus the sum of the previous event probabilities. This could be termed a residual probability. Alternatively, using a series of *relative* likelihood judgments in the first instance means all information on all events is utilized in deriving the initial probability distribution.

Each event has certain attributes, properties, factors, causes, etc. that interact and in totality determine its likelihood in the eyes of the DM. The second event in the comparison possesses similar attributes, etc., possibly to different degrees, leading to a relative gain or loss in likelihood over the first event. In the absence of a standard measuring device for likelihood, the less likely event in the comparison serves as the “standard”. The more likely event is then assigned a value greater or equal to one (e.g., 1.1 or 10% greater) according to its assessed greater or equal likelihood. Apart from the least and most likely events, each event is compared against two other events ensuring some corroboration of its intrinsic likelihood. And in a complete AHP analysis as above, each event is pairwise compared against every other event.

It may be envisaged theoretically that it is possible to delineate all such attributes, causes, etc. and analytically assess their individual importance for a likelihood estimate for each event. This could be a full AHP analysis with a sophisticated hierarchy of cause and effect leading to resulting probabilities for the respective events. The methodology presented here presumes this to be beyond the capabilities of the DM but that he/she can intuitively synthesize this myriad of causations into an overall percentage gain in likelihood for one event over another in the pairwise comparison. Hence there results a percentage judgment that one event is 10% (slightly), 50% (significantly), or 90%+ (substantially) “more likely” than the other event.

Even if the DM uses an incomplete and/or incorrect likelihood assessment framework for these “more likely” judgments, he/she may fortuitously approach the correct *relative* likelihoods in a series of pairwise comparisons resulting in a “correct” initial probability distribution. That is, faulty or incomplete reasoning can still be useful if it is applied consistently.

Using the current one-event-at-a-time methodology but with the same cognitive limitations, DMs must undertake the harder task of estimating probabilities directly. The methodology outlined here requires similar but less demanding judgments in the first instance.

Conclusions

Taleb (2005) has written that “considering that alternative outcomes could have taken place, that the world could have been different, is the core of probabilistic thinking”(p. x Preface). This point proved valuable for PIMCO (a fixed income fund manager) that considered a disorderly liquidation of Lehman Brothers possible, but gave that scenario the lowest probability of the three scenarios it envisaged (El-Erian, 2016). The other two scenarios were a takeover by a stronger bank (highest probability) and an orderly liquidation similar to what occurred with Long Term Capital Management in 1998. Due to its perceived possibility and resulting consideration by PIMCO, the fund was able to move swiftly when the lowest probability scenario did in fact materialize. Procedures such as those illustrated here may stimulate DMs to think probabilistically and prepare themselves for whatever alternative eventuates.

Estimating probabilities of uncertain events or scenarios one at a time as done traditionally is a flawed process. This approach means that the last event in the procedure has its probability predetermined at one minus the sum of the probabilities of the previously assessed events (a residual probability).This could mean a series of reassessments may be necessary before a final probability distribution is arrived at.

The procedure outlined here requires (n- 1) pairwise judgments for adjacent events in an n-event problem. A full AHP analysis requires $n(n$- 1)/2 judgments. That is, 9 and 45 judgments respectively for a 10-event problem. The additional judgments could improve the “correctness” of the initial probability distribution or could prove redundant if (almost) perfectly consistent with the initial (n- 1) adjacent event pairwise values.

Imposing a structure on the process as outlined here (first an ordering and secondly a series of pairwise or primary judgments) means no probability is determined as a residual probability. Also,any series of distribution reassessments is avoided, apart from any final judgments by the DM as illustrated above. Furthermore, in deriving the initial or “ballpark” probability distribution, *all* information on *all* events is utilized in the first instance.And all calculations can be completed by a numerate layperson on the back of an envelope with only a calculator.

References

Brunelli, M. (2015). *Introduction to the analytic hierarchy process.* Heidelberg: Springer.

Crawford, G.B. (1987). The geometric mean procedure for estimating the scale of a judgment matrix. *Mathematical Modelling, 9*(3-5), 327-334.

El-Erian, M.A. (2016). *The only game in town.* New York: Random House.

Hughes, W.R. (2009). A structural framework for strategic decision making with AHP: Probability assessment and Bayesian revision. *Omega, 37*(2), 463-470.

Hughes, W.R. (2010).Assessing probabilities of unique events: Calibrating qualitative judgments into a probability distribution. *Chinese Business Review, 9*(9), 54-60.

Saaty, T.L. (1980). *The analytic hierarchy process.* New York: McGraw-Hill.
Saaty, T.L. (2005). *Theory and applications of the analytic network process.* Pittsburgh: RWS Publications.
Taleb, N.M. (2005). *Fooled by randomness: The hidden role of chance in life and in the markets*(2nd ed.). New York: Random House.