Chinese Business Review, Jan-Feb. 2022, Vol. 21, No. 1, 1-4
doi: 10.17265/1537-1506/2022.01.001

Thinking Probabilistically Revisited

Warren Richard Hughes
University of Waikato, Hamilton, New Zealand

Probability distributions are derived for real-world situations where the environment may be subject to high volatility involving radical revisions in probability judgments. A simple procedure is outlined deriving an initial probability distribution which may then be adjusted to reflect additional or new information. The trade-off between minimal computation and maximum information is examined.

Keywords: probability assessment, pairwise judgments, reciprocal matrices, eigenvalues, geometric means

Introduction

The very simple procedure as in Hughes (2020; 2021) is reviewed showing how to quantify beliefs about possible events or scenarios when initially only rudimentary or tentative ideas about the relative likelihoods have been formulated. As a first step the decision-maker (DM) orders the possible events from least to most likely. Then a series of pairwise judgments for typical events X and Y are made by the DM qualitatively with suggested associated quantitative values as follows:

- X and Y are equally likely (1.0);
- X is a little more likely than Y (1^+-1.25 or average 1.13^+);
- X is not quite twice as likely as Y (1.75-2^- < 2 or average 1.88^-);
- X is between two and three times more likely than Y (2.5).

Of course, quantifying qualitative ideas accurately takes practice. One way is to average a range of values as above. Say you believe X is a "little more" likely than Y. But should "little more" be 10% or 40% more likely? Using an X/Y ratio of (1.1 + 1.4)/2 or 1.25 will get you to the "ballpark". Over time, exactly what a "little more" means to you can be refined more precisely.

Possibly values such as 1.15, 1.2, 1.25 etc. will seem more natural to the DM than say 1.13 as above. Whatever values are used by the DM, they may only be vague or rudimentary and the approximations as above may be useful. Pairwise judgments, however, are only a means to the end of a "ballpark" distribution and not fixed points to be adhered to.

Structuring Probability Assessments

To illustrate the methodology, take a situation with four events A, B, C, and D and allow this ordering to be from least to most likely in the DM's view. Using the illustrative pairwise values above, the probabilities can be determined as in the following table, with A least likely (base value 1.0) and D most likely with the likelihood of D over C (D/C) judged to be 2.5.

Warren Richard Hughes, Dr., honorary fellow, Department of Economics, University of Waikato, Hamilton, New Zealand.
Correspondence concerning this article should be addressed to Warren Richard Hughes, 6 Taurarua Terrace, Parnell, Auckland, New Zealand 1052.

Table 1

Computation of the Probabilities From the Initial Pairwise Judgments

Event	Pairwise value	Compound value	Probability	Percent	ML % P(·)*
A	1.00	1.00	1.00/9.5654 = 0.105	10%	Base = 1.0
B	1.13	1.00 x 1.13 = 1.13	1.13/9.5654 = 0.118	12%	12/10 = 1.20
C	1.88	1.13 x 1.88 = 2.1244	2.1244/9.5654 = 0.222	22%	22/12 = 1.83
D	2.50	2.1244 x 2.50 = 5.311	5.311/9.5654 = 0.555	56%	56/22 = 2.55
Totals		9.5654	1.000	100%	

Note. * More likely values for percentage probabilities.

Compound values are easily determined as in Column 3 with the C/A value necessarily 2.1244 and the D/A value at 5.311 as dictated by the preceding pairwise values. Probabilities are then determined as in Column 4 and percentage probabilities follow. In Column 5 we show the "more likely" (ML) values based on the percentage probabilities. Differences from the initial pairwise values are not material to the analysis, unless substantially at variance with the DM's re-considered judgments. The initial pairwise judgments are only a means to the end of a "ballpark" distribution which can then be altered as illustrated below following Table 3.

Table 2

Probabilities Using the Principal Eigenvector Method

Event	Pairwise judgment	Perfectly consistent reciprocal matrix				Priorities or Eigenvalues	Probabilities	Percent probabilities
		A	B	C	D			
A	1.00	1	1/1.13	1/2.1244	1/5.311	0.169039	0.105	10%
B	1.13	1.13	1	1/1.88	1/4.7	0.191014	0.118	12%
C	1.88	2.1244	1.88	1	1/2.5	0.359106	0.222	22%
D	2.50	5.311	4.7	2.5	1	0.897765	0.555	56%
Totals						1.616924	1.000	100%

Constructing a reciprocal matrix as in Table 2 (with appropriately consistent values for the non-adjacent pairwise values as in 1.88 x 2.5 = 4.7 for C/B · D/C = D/B) and using the principal eigenvector methodology as outlined in Saaty (2005), the same probabilities as in Table 1 are derived. Crawford (1987) outlined an alternative method for deriving probabilities (or priorities in analytic network/hierarchy process terminology) using the geometric mean. Typically, there are very small differences in the probabilities derived with either method. A modern review of both procedures (and others) is given by Brunelli (2015).

Pairwise Judgments on All Events

It is possible that construction of the perfectly consistent reciprocal matrix as in Table 2 triggers further judgments of the DM for the non-adjacent events. Specifically let us suppose that C/A becomes 2.0 (C's greater likelihood over A is now slightly reduced), D/A reduces to 4.0, and D/B reduces to 3.0 (D's predominance over A and B also now reduced). Again, it might seem more natural for a DM to use integers here or alternatively a value like 3.5 to reflect a judgment of between three and four times more likely. Resulting probabilities using the eigenvector and geometric mean methodologies are shown in Table 3.

Note first that probabilities derived using either the principal eigenvector or geometric mean method are very close and identical when expressed in percentage terms. As foreshadowed by the re-considered judgments, the probability of D drops by 6% and the probabilities for all other events rise slightly. Re-consideration by the

DM may have resulted in a recognition that the probabilities of the less likely events A, B, and C were too small when considered as a group. That is, there was a 50% chance that one of them could occur. This illustrates one advantage of using all n(n-1)/2 judgments in a n-event problem as opposed to the minimal (n-1) judgments. Of course, the DM could have made these adjustments directly following the Table 1 results, although implications from the additional pairwise judgments may be useful before finalizing the distribution.

Table 3

Pairwise Judgments on All Events

Event	Reciprocal matrix				Priorities or eigenvalues	Geometric mean	Probabilities		
	A	B	C	D			Eig' values	Geo mean	Percent
A	1	1/1.13	1/2	1/4	0.209607	0.576711	0.1223	0.1229	12%
B	1.13	1	1/1.88	1/3	0.244354	0.669037	0.1425	0.1425	14%
C	2.0	1.88	1	1/2.5	0.404605	1.107419	0.2360	0.2360	24%
D	4.0	3.0	2.5	1	0.855950	2.340347	0.4992	0.4986	50%
Totals					1.714516	4.693514	1.0000	1.0000	100%

Further adjustments such as accounting for probability differences are also possible. When considering the distribution in Table 3, the DM may determine that event A is at most a 10% chance. This allows event B to be increased to a 15% chance with a 5% differential over A leaving C with a 10% differential over B with its increased probability to 25%. In this case, the probability of one of A, B, or C occurring remains at 50% and this could be an additional (non-pairwise) assessment of the DM. Judgments such as these on probability differences between events are now easier to make with an axiomatically correct distribution as the starting point.

Conclusions

As demonstrated above, pairwise judgments are not the only input into the DM's final distribution. And even for the pairwise judgments it could be assumed that a set of common factors pertains to each pairwise judgment in turn. In reality, it may be that a certain pairwise judgment highlights factors that may be missing (or of lesser effect) from the other pairwise judgments. Should this be of concern? Possibly not with the averaging process (either by the eigenvector method or geometric mean) over all pairwise judgments sufficing to incorporate all relevant factors appropriately at some point in determining the "ballpark" distribution. There is a tradeoff between the economy of method with the minimal number of judgments versus ensuring all factors are accounted for (if not to the same extent in every pairwise judgment) in determining the "ballpark" distribution. This could be an argument for requiring a complete reciprocal matrix initially.

In the real world of the Internet-of-Things, relevant information is arriving almost continuously and opinions about the likelihoods of various events in certain situations diverse, widely disseminated, and subject to volatility. The origin of the COVID-19 virus is but one example. This almost continuous arrival of information relevant to certain situations makes revision of probability calculations essential. Here the minimal (n-1) pairwise comparisons for a n-event problem seems optimal requiring least computation. On the other hand, since each pairwise judgment may incorporate factors unique to that particular assessment, n(n-1)/2 judgments over all events make use of all information and may therefore serve to make the initial "ballpark" distribution more "correct".

References

Brunelli, M. (2015). *Introduction to the analytic hierarchy process.* Heidelberg: Springer.

Crawford, G. B. (1987). The geometric mean procedure for estimating the scale of a judgment matrix. *Mathematical Modelling, 9*(3-5), 327-334.

Hughes, W. R. (2020). Structuring probability assessments. *Chinese Business Review, 10*(5), 171-177.

Hughes, W. R. (2021). Thinking probabilistically. *Chinese Business Review, 20*(5), 164-167.

Saaty, T. L. (2005). *Theory and applications of the analytic network process.* Pittsburgh: RWS Publications.