DECISION TREES TO DERIVE FAIR PRICES IN UNIQUE SITUATIONS UNDER UNCERTAINTY

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ABSTRACT

Decision trees can be used to detail a transparent derivation of a "fair" price for an asset, security etc. (Fair Price) where various outcomes are possible leading to differing final prices on completion of the transaction. A final branch of the tree or endpoint will show the dollar value realized should that path be the one followed. This paper outlines a new procedure for determining the probability of that endpoint being realized where the situation under analysis is unique. That is, subjective probabilities need to be calculated based only on the judgment of the decision-maker since historical frequencies are not pertinent to the unique situation. Procedures outlined here make probability assessment easier for business professionals by minimizing the mathematical input required. Once all probabilities are determined, a Fair Price or expected value can be calculated for the asset, security, action etc. under analysis. The probability deriving procedure can also be utilized in a Bayesian Revision process to revise the Fair Price following news or information relevant to the unique situation. The procedure is illustrated using an example showing the derivation of a Fair Price for a share in a company subject to an unsolicited takeover offer but can be utilized in other decision situations.

INTRODUCTION

Business situations can occur that require a step-by-step analysis (possibly through time) where the end result could be one of a number of possible outcomes. Decision trees have proven to be a useful device in mapping the paths to the possible endpoints, Lindley (2006), Jensen and Nielsen (2007). One difficulty with such decision trees is the determination of probabilities for branches emanating from a random or chance node or event in the tree. For many business professionals with only rudimentary ideas on probability assessment, this could be an insurmountable problem rendering decision trees of limited value. This paper outlines a procedure for determining probabilities that decision makers (DMs) could follow which does not require extensive mathematical technique. This makes decision tree methodology readily available to DMs where chance or random nodes form part of the tree. DMs are, however, assumed to possess knowledge on the relative likelihoods of the branches emanating from a chance node. Judgments on these relative likelihoods will have been formed from past experience in similar situations possibly unique to the DM. Qualitative judgments of the DM on the relative likelihoods of possible outcomes from chance nodes can then be rendered precise using the procedure outlined here.

The procedures are outlined using an illustrative example for determining a fair price for the stock of a company subjected to an unsolicited takeover offer.

PROBABILITY ASSESSMENT

Where part of the decision tree involves an event subject to uncertainty (a random or chance node), the possible outcomes or branches emanating from that node will typically exhibit different likelihoods of occurrence. The DM is assumed to possess qualitative judgments (see Table 1 below) as to these relative likelihoods enabling him/her to rank the outcomes in order from least likely to most likely. The least likely outcome is assigned a likelihood of unity. The next least likely outcome must then be rated as to how much "more likely" it is relative to the least likely (immediately preceding) outcome in the ranking. For example, a relative likelihood pairwise value of 1.33 for the second least likely outcome indicates this outcome to be one third "more likely" than the least likely outcome. This judgment of 1.33 would be clearly subjective and may be idiosyncratic of an individual DM. In deciding on the definitive pairwise likelihood value, it may be advantageous for the DM to also consider the reciprocal value of the current "more likely" judgment. This gives the likelihood for the less likely outcome as a percentage of the likelihood of the more likely outcome. For example, an initial pairwise "more likely" value of 1.33 (4/3) means the less likely outcome has only 75% (3/4) of the likelihood of the more likely outcome. The DM, on reflection, may consider this to be too low with an 80% (4/5) or 83% (5/6) percentage better reflecting the DM's definitive pairwise judgment. As a result, the "more likely" value used in the calculations would then reduce to 1.25 or 1.20 as the case may be.

The DM progresses through the ranked outcomes making successive pairwise judgments until all outcomes have been compared to the immediately preceding outcome in the ranking. For any pairwise judgment, any relative frequency information available from past history concerning the two outcomes could be employed in making the definitive "more likely" judgment. The use of pairwise judgments in decision making was initially developed by Saaty (1980) leading to the Analytic Hierarchy Process (AHP) and has been most recently expounded by him in Saaty (2005).

It was noted above how a DM may have qualitative notions on the relative likelihoods of two outcomes or branches under comparison. A procedure for mapping such qualitative judgments into quantitative values must be pragmatic and simple to follow. The first task in assessing pairwise likelihoods is the "order of magnitude". Assume the DM assesses the "more likely" outcome of two possibilities to be less than twice as likely as the other outcome. Following this first judgment, the next step is to assess the extent of the "more likely" magnitude as one of equally likely, slightly more likely, significantly more likely or substantially more likely. Table 1 below outlines a suggested range of quantitative pairwise likelihood values for each of these qualitative judgments.

Table 1

Suggested Nomenclature For Quantifying Qualitative Pairwise Likelihood Judgments

Qualifier of "likely" in pairwise comparison of likelihood for the "more likely" outcome	Pairwise Likelihood Range	Mid-Point
Equally	1.0	1.00
Slightly more	1.0^{+} - 1.2^{-}	1.10
Significantly more	1.2 - 1.8^{-}	1.50
Substantially more – just less than twice as	1.8 - 2.0^{-}	1.90
Twice as	2.0	2.00

If the DM initially determines the order of magnitude to be two to three times "more likely", the mid-points of the three appropriate ranges then become 2.10, 2.50 and 2.90 respectively. Ranges for higher orders of magnitude are similarly defined. While the mid-point values are suggestive for the qualitative qualifiers slightly, significantly and substantially, the DM should select a specific value in the appropriate range for the final "more likely" judgment.

After the pairwise likelihoods have been determined for the ranked outcomes, the compound likelihood for each possibility is derived as the product of all previous pairwise values up to the outcome currently being assessed. Each outcome then can be associated with a compound likelihood which constitutes **the** likelihood of that outcome or branch of the tree. Normalizing these likelihoods results in the probability distribution over all outcomes or branches based on the preceding pairwise judgments. After appraising the complete probability distribution derived from the separate pairwise judgments between adjacent (in likelihood) outcomes, the DM may fine-tune the probabilities to better reflect his/her final assessment of the relative likelihoods. The procedures summarized above are explained more fully in Hughes (2010), including procedures for group assessment of the pairwise likelihoods.

BAYESIAN REVISION

Through time, news or information on the situation under analysis may become available. At the time of the news arrival, the situation may have evolved to a particular point on the tree involving uncertain outcomes before proceeding to the next point or possibly an endpoint. Under the preceding analysis, a probability distribution over the possible outcomes will have been assessed. This constitutes the prior distribution in Bayesian terminology. Given the news or information just to hand, all possible outcomes from the random or chance node would be ranked from least to most "consistent with the news or information". Some outcomes may have zero consistency with the news and this will result in their subsequent elimination from the analysis (see below). The remaining outcomes or branches are then ranked from least to most consistent with the news and pairwise likelihoods assigned accordingly exactly as outlined in the above procedure for the initial probability distribution. The resulting compound likelihood values starting at 1.0 comprise the "news" likelihoods over all remaining possibilities leading to joint (prior x likelihood) values and the consequent posterior distribution over all remaining outcomes. Use of this posterior distribution in place of the previous prior distribution will result in a revised Fair Price subsequent to the news arrival.

VALIDATION OF DERIVED PROBABILITIES

The above procedure results in probability distributions (either initially or after Bayesian Revision) over all possible outcomes emanating from random or chance nodes in the tree. These distributions reflect the DM's personal and subjective judgments. To validate these judgments to some degree, some other input is required independent of the original DM. One possibility is to take the "most likely" outcome as currently determined by the DM at some random node and seek an odds judgment from another expert on the situation under analysis as to the likelihood of this outcome obtaining to the exclusion of all other possible outcomes at that node. The situation is akin to picking the winner of a horserace and requires only one input by the expert and not a complete probability distribution. If the resulting expert's odds are "close to" the odds derived from the DM's distribution, the procedure leading to the current probability distribution can be said to be "not invalidated". Odds significantly at variance with the DM's judgments should prompt a re-evaluation by the DM of his/her probability distribution over all outcomes at that node. More than one expert could be so consulted to provide increased validation or otherwise. Furthermore, the "most likely" outcome could be replaced by a lesser likely but more "focal" outcome that may be the subject of more readily available odds from other experts due to its prominence in the public consciousness (e.g. a double dip recession, sustained GDP growth at 5%+ p.a. etc.)

ILLUSTRATIVE DECISION TREE EXAMPLE TO DETERMINE A FAIR PRICE

The procedures outlined above are illustrated using a business example of a company subjected to an unsolicited takeover offer. The situation and resulting possible paths to an endpoint are summarized in an associated EXCEL spreadsheet (available on request from the author) in the form of a decision tree.

Stage 1

Suppose the share price of the target company (TC) immediately prior to the offer was $25. The potential acquiring company (AC) is offering $30 per share. Table 2 shows the Stage 1 options available to TC with illustrative values for the share price if a particular option is taken up.

Table 2

Stage 1 Options For The Target Company After Initial Offer Of $30

Options	Price per Share
Accept initial offer of AC (Accommodate)	$30
Fight for higher price from AC (Hostile)	$32
Seek another partner	Stage 2

Seeking another partner could be a tactic for the Hostile option as well and there may be other genuine partners prepared to offer a price higher than $32 as will be assumed here. The Stage 2 options are shown below.

Stage 2

It is assumed for simplicity that there are two potential partners X and Y (other than AC) that could be prepared to offer TC a higher price per share. Any such merger will, however, be subject to regulatory oversight. Such oversight is assumed (for simplicity) to result in one of three outcomes: Veto, merger allowed (OK) and merger allowed with conditions (OK with Conditions). Illustrative dollar outcomes for TC's share value for both partners and the three possibilities are summarized in Table 3.

Table 3

Stage 2 Outcomes For The TC With Either Partner X Or Y

Stage 2 Options	**Price per Share**
Partner X	
OK	$38
OK with Conditions	$35
Veto	$30
Partner Y	
OK	$35
OK with Conditions	$32
Veto	$30

All stock prices other than the $30 offer price can be considered as either point estimates or as expected values derived from a possible distribution over a range of prices, simulation or other process.

The stock prices received by TC summarized in Table 3 show Partner X to be the more valuable option possibly reflecting existing trade arrangements, mutual shareholdings, better geographic mix etc. Of course, this may increase the probability of regulatory conditions being imposed or possibly a veto on a merger of X with TC. If a veto does occur for either partner, TC's share price is assumed to revert to the current offer price of $30.

Probability Assessments

The probability assessments using the suggested mid-point values from Table 1 are summarised in Table 4.

Table 4

Probability Assessments For Stage 1 Options

Stage 1 Options	Pairwise Judgment	Pairwise Likelihood	Compound Likelihood	Probability
Accommodate	Least likely outcome	1.0	1.0	1/6 = 0.1667
Hostile	Twice as likely as Accommodate	2.0	2.0	2/6 = 0.3333
Seek Partner	Significantly more likely than Hostile	1.5	3.0	3/6 = 0.5000
TOTALS			**6.0**	**1.0000**

Takeover situations will have unique characteristics depending on the parties involved. The qualitative Pairwise Judgments shown in Table 4 are illustrative only and are not meant to represent a norm. Illustrative assessed probabilities for the Stage 2 outcomes with partners X and Y are summarized in Table 5.

Table 5

Illustrative Pairwise Likelihoods And Probability Assessments For Stage 2 Outcomes

Stage 2 Options	Pairwise Judgment	Pairwise Likelihood	Compound Likelihood	Probability
Partner X				
OK	Least likely outcome with X	1.0	1.0	0.1852
OK with Conditions	Twice as likely as previous or OK	2.0	2.0	0.3704
Veto	Slightly more likely than previous	1.2	2.4	0.4444
TOTALS			**5.4**	**1.0000**
Partner Y				
Veto	Least likely outcome with Y	1.0	1.0	0.1429
OK with Conditions	Significantly more likely than Veto	1.5	1.5	0.2143
OK	3 x more likely than OK with Conditions	3.0	4.5	0.6428
TOTALS			**7.0**	**1.0000**

The resulting expected values for a merger with Partner X or Y are calculated using the above probabilities and share prices in Table 3 as shown in the associated spreadsheet available from the author:

Expected Value of merger with Partner X = $33.33

Expected Value of merger with Partner Y = $33.64

For a 50% chance that either Partner could be chosen, the expected value of seeking a partner for TC as at Stage 1 is $33.49. The slightly higher expected value for Partner Y reflects the significantly higher probability of a veto with Partner X than for Partner Y. Based on all the preceding judgments, a Fair Price for TC immediately following the AC offer of $30 a share would be $32.41 as detailed in the associated spreadsheet available from the author.

Bayesian Revision Following Suggested Merger with Partner Y

To continue with the illustrative example, we now suppose that Partner Y has emerged as the favored partner for a merger with TC. Without spelling it out in any great detail, it is assumed that news on this potential merger has eventuated based on informed analyst opinion that makes a veto by the regulatory authority extremely unlikely and that an unconditional merger of TC with Y is the most likely outcome. The resulting illustrative Bayesian revision calculations are summarized in Table 6.

Table 6

Bayesian Revision Following News That Unconditional Merger With Y Is Most Likely

Possible Outcomes	Prior	Pairwise Likelihood	Compound Likelihood	Joint	Posterior	Outcome Prices	Expected Value
Veto	0.1429	1.0	1.0	0.1429	0.0066	30	0.1987
OK w Conditions	0.2143	10.0	10.0	2.1430	0.0994	32	3.1792
OK	0.6428	3.0	30.0	19.284	0.8940	35	31.2908
TOTALS	**1.0000**		**41.0**	**21.5699**	**1.000**		**34.6687**

The prior distribution is reproduced from the pre-news analysis in Table 5. In this illustrative example, the ranking of alternative outcomes as to consistency with the "news" follows exactly the prior probability ranking from least to most likely. That is, the "news" has reinforced our prior beliefs. This need not necessarily be the case. As noted above, the new information may be inconsistent with events or states previously thought possible. For example, we may learn that a new appointment to a position from a field of possible candidates will be of Asian ethnicity. All non-Asian candidates then have zero consistency with this knowledge. In general, events having zero consistency with the new information are ranked first in the Bayesian Revision calculation and assigned a zero likelihood of news consistency. Of the remaining events, the event least consistent with the new information is assigned a Pairwise Likelihood of 1.0. Subsequent events are ranked according to increased (or equal) consistency with the new information. Pairwise Likelihoods are then assigned as previously according to the degree of increased (or equal) consistency with the new information over the preceding ranked alternative. From these Pairwise Likelihoods, the Compound Likelihoods for each event are then calculated (starting with the event of least consistency at 1.0) the result being the likelihood of that event given the new information. The prior probabilities derived previously are then allocated appropriately to the newly ranked events with a resulting joint likelihood and posterior probability derived in the usual way. Of course, any events having zero consistency with the new information will have a zero posterior probability.

In the example of Table 6, the Pairwise Likelihood of 10 for **OK with Conditions** reflects the view that a **Veto** here is extremely unlikely and not that **OK with Conditions** is extremely likely. In fact, **OK** is regarded as 3 times more likely than **OK with Conditions**. The Compound Likelihoods become **the** likelihoods for the possible outcomes in Bayesian terminology and the Joint is the product of the Prior column with this column (i.e. Prior x Likelihood). The Posterior normalizes the Joint column into a probability distribution and the resulting Fair Price based on this distribution is $34.67 which is just below Y's maximum price of $35. This reflects the assessment of only a very small chance (about 10%) that a final price lower than $35 will eventuate. Note that if we changed the Pairwise Likelihood value 10 to 20 in Table 6 above, the resulting Fair Price would be an expected value of $34.68, one cent greater than the current $34.67. This suggests there is no real gain from trying to fine-tune the second Pairwise Likelihood in Table 6 to a more appropriate value between 10 and 20. The order-of-magnitude value 10 that effectively minimizes the **Veto** probability suffices for a ball-park Fair Price derivation. It should be noted that use of a high order-of-magnitude value such as 10 at any outcome effectively minimizes the probabilities of **all** preceding outcomes in the likelihood ranking. Due to the low posterior probability for **Veto**, the post-veto share price (assumed to be $30 here) is also not an important input into the Fair Price derivation in this example. Note that the posterior probability for an **OK** outcome implies odds of just under 9: 1 on. Odds of say 7:1 on out to 10:1 on from other experts for this event could be said to "not invalidate" the DM's judgments in this example.

The above example demonstrates how Bayesian revision can be undertaken where "news" can alter previous judgments although such "news" may not definitively determine what the final result may be. In this particular case, however, the DM could decide to ignore all previous analysis and analyse the current situation (merger of TC with Y) as a standalone problem. The final price per share for TC could, of course, be subject to some uncertainty. In this situation, the Prior above becomes redundant and all analysis is based only on the information relating to a merger of TC with Y. The Bayesian analysis above could still be applicable if a merger of TC and Y was considered the most likely outcome by business analysts without any confirmation by TC and Y. In this case, the information reflected in the Prior distribution could still be relevant.

An example of Bayesian revision in another context is contained in Hughes (2009).

CONCLUSIONS

The probability assessment procedure presented here may have advantages for modern business problem solving and strategic planning. Many business problems will involve unique situations making probability assessment over the relevant outcomes also unique. If business analysts have judgments about the relative likelihoods of the alternative outcomes, the procedure presented here is one way of transforming rudimentary qualitative judgments into quantitative probabilities. Using decision trees in the form of standard spreadsheets makes the "what if" calculations easy to generate with a resulting range of prices, expected values etc. for sensitivity analysis. Making probability assessments and Bayesian revision more routine by using procedures outlined here (or similar easy-to-use methods) and the ability to generate "what if" alternatives with these assessments (and other inputs) makes the use of decision trees an invaluable tool for calculating Fair Prices or expected values of actions. Accordingly, decision trees using spreadsheets will continue to be a valuable guide to informed valuation, strategic analysis or decision making under uncertainty generally.

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